

# VANISHING OF COHOMOLOGY OVER HYPERSURFACE RINGS

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ABSTRACT. Let  $R$  be a hypersurface ring and let  $M$  and  $N$  be  $R$ -modules. It is shown that the vanishing of  $\text{Ext}_R^i(M, N)$  for a certain number of consecutive values of  $i$  starting at  $n$  forces the complete intersection dimension of  $M$  to be at most  $n - 1$ . We also estimate the complete intersection dimension of  $M^*$ , the dual of  $M$ , in terms of vanishing of the cohomology modules,  $\text{Ext}_R^i(M, N)$ .

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## 1. INTRODUCTION

In this paper, we study the relationship between the vanishing of  $\text{Ext}_R^i(M, N)$  for various consecutive values of  $i$ , and the complete intersection dimensions of  $M$  and  $M^*$ , the dual of  $M$ . The vanishing of homology was first studied by Auslander [3]. For two finitely generated modules  $M$  and  $N$  over an unramified regular local ring  $R$ , he proved that if  $\text{Tor}_i^R(M, N) = 0$  for some  $i > 0$  then  $\text{Tor}_n^R(M, N) = 0$  for all  $i \geq n$ . In [20], Lichtenbaum settled the ramified case. It is easy to see that a similar statement is not true in general, with Tor replaced by Ext. In [18], Jothilingam studied the vanishing of cohomology by using the rigidity Theorem of Auslander. For two non-zero modules  $M$  and  $N$  over a regular local ring  $R$ , he proved that if  $M$  satisfies  $(S_n)$  for some  $n \geq 0$  and  $\text{Ext}_R^i(M, N) = 0$  for some positive integer  $i$  such that  $i \geq \text{depth}_R(N) - n$  then  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$ . In [19], Jothilingam and Duraivel studied the relationship between the vanishing of  $\text{Ext}_R^i(M, N)$  and the freeness of  $M^*$ . For two non-zero modules  $M$  and  $N$  over a regular local ring  $R$ , they proved that if  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - 2\}$  then  $M^*$  is free. In this paper we are going to generalize these results.

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An  $R$ -module  $M$  is said to be  $c$ -rigid if for all  $R$ -modules  $N$ ,  $\text{Tor}_{i+1}^R(M, N) = \text{Tor}_{i+2}^R(M, N) = \cdots = \text{Tor}_{i+c}^R(M, N) = 0$  for some  $i \geq 0$  implies that  $\text{Tor}_n^R(M, N) = 0$  for all  $n > i$ . If  $c = 1$  then we simply say that  $M$  is rigid.

The aim of this paper is to study the following question.

**Question 1.1.** Let  $R$  be a Gorenstein local ring and let  $M$  and  $N$  be  $R$ -modules such that  $N$  has reducible complexity. Assume that  $n \geq 0$ ,  $c > 0$  are integers and that  $N$  is  $c$ -rigid. If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{c, \text{depth}_R(N) - n\}$  then what can we say about the Gorenstein dimensions of  $M$  and  $M^*$ ?

In section 2, we collect necessary notation, definitions and some known results which will be used in this paper.

In section 3, we study the Question 1.1 for rigid modules. Over a Gorenstein local ring  $R$ , given non-zero  $R$ -modules  $M$  and  $N$  such that  $N$  has reducible complexity, we show that if  $N$  is rigid and  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{G-dim}_R(M^*) \leq n - 2$ . Which is a generalization of [19, Theorem 1]. Also it is shown that if  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N)\}$  then  $\text{G-dim}_R(M) = 0$  (see Theorem 3.2). As a consequence, for two non-zero modules  $M$  and  $N$  over an admissible hypersurface ring  $R$  with isolated singularity, it is shown that if  $[N] = 0$  in  $\overline{G}(R)_{\mathbb{Q}}$  and  $\text{Ext}_R^i(M, N) = 0$  for some positive integer  $i$  such that  $i \geq \text{depth}_R(N)$  then  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$  (see Corollary 3.10).

In section 4, we study the Question 1.1 for reflexive modules over hypersurfaces. For two modules  $M$  and  $N$  over a hypersurface ring  $R$ , It is shown that if  $N$  is reflexive and has constant rank and  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N)\}$  then  $\text{CI-dim}_R(M) = 0$  (see Theorem 4.3). Also it is shown that if  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N) - 2\}$  then either  $M^*$  is free or  $\text{pd}_R(N) < \infty$  (see Theorem 4.4).

## 2. PRELIMINARIES

Throughout the paper,  $(R, \mathfrak{m})$  is a commutative Noetherian local ring and all modules are finite (i.e. finitely generated)  $R$ -modules. The codimension of  $R$  is defined to be the non-negative integer  $\text{embdim}(R) - \dim(R)$  where  $\text{embdim}(R)$ , the embedding dimension of  $R$ , is the minimal number of generators of  $\mathfrak{m}$ . Recall that  $R$  is said to be a complete intersection if the  $\mathfrak{m}$ -adic completion  $\hat{R}$  of  $R$  has the form  $Q/(f)$ , where  $f$  is a regular sequence of  $Q$  and  $Q$  is a regular local ring. A complete intersection of codimension one is called a hypersurface. A local ring  $R$  is said to be an admissible complete intersection if the  $\mathfrak{m}$ -adic completion  $\hat{R}$  of  $R$  has the form  $Q/(f)$ , where  $f$  is a regular sequence of  $Q$  and  $Q$  is a power series ring over a field or a discrete valuation ring. Let

$$\cdots \rightarrow F_{n+1} \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

be the minimal free resolution of  $M$ . Recall that the  $n^{\text{th}}$  syzygy of an  $R$ -module  $M$  is the cokernel of the  $F_{n+1} \rightarrow F_n$  and denoted by  $\Omega^n M$ , and it is unique up to isomorphism. The  $n^{\text{th}}$  Betti number, denoted  $\beta_n^R(M)$ , is the rank of the free  $R$ -module  $F_n$ . The complexity of  $M$  is defined as follows.

$$\text{cx}_R(M) = \inf\{i \in \mathbb{N} \cup 0 \mid \exists \gamma \in \mathbb{R} \text{ such that } \beta_n^R(M) \leq \gamma n^{i-1} \text{ for } n \gg 0\}.$$

Note that  $\text{cx}_R(M) = \text{cx}_R(\Omega^i M)$  for every  $i \geq 0$ . It follows from the definition that  $\text{cx}_R(M) = 0$  if and only if  $\text{pd}_R(M) < \infty$ . If  $R$  is a complete intersection, then

the complexity of  $M$  is less than or equal to the codimension of  $R$  (see [15]). The complete intersection dimension was introduced by Avramov, Gasharov and Peeva [7]. A module of finite complete intersection dimension behaves homologically like a module over a complete intersection. Recall that a quasi-deformation of  $R$  is a diagram  $R \rightarrow A \leftarrow Q$  of local homomorphisms, in which  $R \rightarrow A$  is faithfully flat, and  $A \leftarrow Q$  is surjective with kernel generated by a regular sequence. The module  $M$  has finite complete intersection dimension if there exists such a quasi-deformation for which  $\mathrm{pd}_Q(M \otimes_R A)$  is finite. The complete intersection dimension of  $M$ , denoted  $\mathrm{CI-dim}_R(M)$ , is defined as follows.

$$\mathrm{CI-dim}_R(M) = \inf\{\mathrm{pd}_Q(M \otimes_R A) - \mathrm{pd}_Q(A) \mid R \rightarrow A \leftarrow Q \text{ is a quasi-deformation}\}.$$

The complete intersection dimension of  $M$  is bounded above by the projective dimension,  $\mathrm{pd}_R(M)$ , of  $M$  and if  $\mathrm{pd}_R(M) < \infty$  then the equality holds (see [7, Theorem 1.4]). Every module of finite complete intersection dimension has finite complexity (see [7, Theorem 5.3]).

The concept of modules with reducible complexity was introduced by Bergh [8]. Let  $M$  and  $N$  be  $R$ -modules and consider a homogeneous element  $\eta$  in the graded  $R$ -module  $\mathrm{Ext}_R^*(M, N) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}_R^i(M, N)$ . Choose a map  $f_\eta : \Omega_R^{|\eta|}(M) \rightarrow N$  representing  $\eta$ , and denote by  $K_\eta$  the pushout of this map and the inclusion  $\Omega_R^{|\eta|}(M) \hookrightarrow F_{|\eta|-1}$ . Therefore we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^{|\eta|} M & \longrightarrow & F_{|\eta|-1} & \longrightarrow & \Omega^{|\eta|-1} M \longrightarrow 0 \\ & & \downarrow f_\eta & & \downarrow & & \downarrow \parallel \\ 0 & \longrightarrow & N & \longrightarrow & K_\eta & \longrightarrow & \Omega^{|\eta|-1} M \longrightarrow 0. \end{array}$$

with exact rows. Note that the module  $K_\eta$  is independent, up to isomorphism, of the map  $f_\eta$  chosen to represent  $\eta$ .

**Definition 2.1.** The full subcategory of  $R$ -modules consisting of the modules having reducible complexity is defined inductively as follows:

- (i) Every  $R$ -module of finite projective dimension has reducible complexity.
- (ii) An  $R$ -module  $M$  of finite positive complexity has reducible complexity if there exists a homogeneous element  $\eta \in \mathrm{Ext}_R^*(M, M)$ , of positive degree, such that  $\mathrm{cx}_R(K_\eta) < \mathrm{cx}_R(M)$ ,  $\mathrm{depth}_R(M) = \mathrm{depth}_R(K_\eta)$  and  $K_\eta$  has reducible complexity.

By [8, Proposition 2.2(i)], every module of finite complete intersection dimension has reducible complexity. In particular, every module over a local complete intersection ring has reducible complexity. On the other hand, there are modules having reducible complexity but whose complete intersection dimension is infinite (see for example, [10, Corollary 4.7]).

The notion of the Gorenstein (or G-) dimension was introduced by Auslander [2], and developed by Auslander and Bridger in [4].

**Definition 2.2.** An  $R$ -module  $M$  is said to be of  $G$ -dimension zero whenever

- (i) the biduality map  $M \rightarrow M^{**}$  is an isomorphism.
- (ii)  $\mathrm{Ext}_R^i(M, R) = 0$  for all  $i > 0$ .
- (iii)  $\mathrm{Ext}_R^i(M^*, R) = 0$  for all  $i > 0$ .

The Gorenstein dimension of  $M$ , denoted  $\text{G-dim}_R(M)$ , is defined to be the infimum of all nonnegative integers  $n$ , such that there exists an exact sequence

$$0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$$

in which all the  $G_i$  have  $G$ -dimension zero. By [4, Theorem 4.13], if  $M$  has finite Gorenstein dimension then  $\text{G-dim}_R(M) = \text{depth } R - \text{depth}_R(M)$ . By [7, Theorem 1.4],  $\text{G-dim}_R(M)$  is bounded above by the complete intersection dimension,  $\text{CI-dim}_R(M)$ , of  $M$  and if  $\text{CI-dim}_R(M) < \infty$  then the equality holds.

Let  $R$  be a local ring and let  $M$  and  $N$  be finitely generated non-zero  $R$ -modules. We say the pair  $(M, N)$  satisfies the depth formula provided:

$$\text{depth}_R(M \otimes_R N) + \text{depth } R = \text{depth}_R(M) + \text{depth}_R(N).$$

The depth formula was first studied by Auslander [3] for finitely generated modules of finite projective dimension. In [16], Huneke and Wiegand proved that the depth formula holds for  $M$  and  $N$  over complete intersection rings  $R$  provided  $\text{Tor}_i^R(M, N) = 0$  for all  $i > 0$ . In [10], Bergh and Jorgensen generalize this result for modules with reducible complexity over a local Gorenstein ring. More precisely, they proved the following result:

**Theorem 2.3.** [10, Corollary 3.4] *Let  $R$  be a Gorenstein local ring and let  $M$  and  $N$  be non-zero  $R$ -modules. If  $M$  has reducible complexity and  $\text{Tor}_i^R(M, N) = 0$  for all  $i > 0$  then  $\text{depth}_R(M \otimes_R N) + \text{depth } R = \text{depth}_R(M) + \text{depth}_R(N)$ .*

We denote by  $G(R)$  the Grothendieck group of finitely generated modules over  $R$ , that is, the quotient of the free abelian group of all isomorphism classes of finitely generated  $R$ -modules by the subgroup generated by the relations coming from short exact sequences of finitely generated  $R$ -modules. We also denote by  $\overline{G}(R) = G(R)/[R]$ , the reduced Grothendieck group. For an abelian group  $G$ , we set  $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ .

Let  $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$  be a finite projective presentation of  $M$ . The transpose of  $M$ ,  $\text{Tr } M$ , is defined to be  $\text{coker } f^*$ , where  $(-)^* := \text{Hom}_R(-, R)$ , which satisfies in the exact sequence

$$(2.1) \quad 0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{Tr } M \rightarrow 0$$

and is unique up to projective equivalence. Thus the minimal projective presentations of  $M$  represent isomorphic transposes of  $M$ . Two modules  $M$  and  $N$  are called *stably isomorphic* and write  $M \approx N$  if  $M \oplus P \cong N \oplus Q$  for some projective modules  $P$  and  $Q$ . Note that  $M^* \approx \Omega^2 \text{Tr } M$  by the exact sequence (2.1).

The composed functors  $\mathcal{T}_k := \text{Tr } \Omega^{k-1}$  for  $k > 0$  introduced by Auslander and Bridger in [4]. If  $\text{Ext}_R^i(M, R) = 0$  for some  $i > 0$ , then it is easy to see that  $\mathcal{T}_i M \approx \Omega \mathcal{T}_{i+1} M$ .

We frequently use the following Theorem of Auslander and Bridger.

**Theorem 2.4.** [4, Theorem 2.8] *Let  $M$  be an  $R$ -module and  $n \geq 0$  an integer. Then there are exact sequences of functors:*

$$(2.4.1) \quad 0 \rightarrow \text{Ext}_R^1(\mathcal{T}_{n+1} M, -) \rightarrow \text{Tor}_n^R(M, -) \rightarrow \text{Hom}_R(\text{Ext}_R^n(M, R), -) \rightarrow \text{Ext}_R^2(\mathcal{T}_{n+1} M, -),$$

$$(2.4.2) \quad \text{Tor}_2^R(\mathcal{T}_{n+1} M, -) \rightarrow (\text{Ext}_R^n(M, R) \otimes_R -) \rightarrow \text{Ext}_R^n(M, -) \rightarrow \text{Tor}_1^R(\mathcal{T}_{n+1} M, -) \rightarrow 0.$$

For an integer  $n \geq 0$ , we say  $M$  satisfies  $(S_n)$  if  $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{n, \dim(R_{\mathfrak{p}})\}$  for all  $\mathfrak{p} \in \text{Spec}(R)$ . If  $R$  is Gorenstein, then  $M$  satisfies  $(S_n)$  if and only if  $\text{Ext}_R^i(\text{Tr } M, R) = 0$  for all  $1 \leq i \leq n$  (see [4, Theorem 4.25]). In particular,  $M$  satisfies  $(S_2)$  if and only if it is reflexive, i.e., the natural map  $M \rightarrow M^{**}$  is bijective, where  $M^* = \text{Hom}_R(M, R)$  (see [14, Theorem 3.6]).

The following results will be used throughout the paper.

**Theorem 2.5.** *Let  $R$  be a local complete intersection ring and let  $M$  and  $N$  be  $R$ -modules. Then  $\text{Tor}_i^R(M, N) = 0$  for all  $i \gg 0$  if and only if  $\text{Ext}_R^i(M, N) = 0$  for all  $i \gg 0$ . Moreover, if  $R$  is a hypersurface then either  $\text{pd}_R(M) < \infty$  or  $\text{pd}_R(N) < \infty$ .*

*Proof.* See [6, Theorem 6.1] and [6, Proposition 5.12].  $\square$

**Theorem 2.6.** *Let  $R$  be a local ring and let  $M$  and  $N$  be two non-zero  $R$ -modules. If  $\text{Ext}_R^i(M, N) = 0$  for all  $i \gg 0$  and  $\text{G-dim}_R(M) < \infty$  then the following statements hold true.*

- (i) *If  $\text{CI-dim}_R(M) < \infty$  then  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\}$ .*
- (ii) *If  $\text{CI-dim}_R(N) < \infty$  then  $\text{G-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\}$ .*

*Proof.* See [1, Theorem 4.2] and [22, Theorem 4.4].  $\square$

**Theorem 2.7.** *Let  $R$  be a local ring and let  $M$  and  $N$  be  $R$ -modules. If  $\text{CI-dim}_R(M) = 0$  then  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  if and only if  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ .*

*Proof.* First note that  $\text{CI-dim}_R(\text{Tr } M) = 0$  by [22, Lemma 3.3] and  $M \approx \text{Tr } \text{Tr } M$ . Now the assertion is clear by [22, Proposition 3.4].  $\square$

### 3. VANISHING OF EXT FOR RIGID MODULES

We start this section by estimate the Gorenstein dimension of the transpose of  $M$  in terms of vanishing of the cohomology modules,  $\text{Ext}_R^i(M, N)$ .

**Lemma 3.1.** *Let  $R$  be a Gorenstein ring and let  $M$  and  $N$  be non-zero  $R$ -modules. Assume that  $n \geq 0$  is an integer and that the following conditions hold.*

- (1)  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$ .
- (2)  $N$  is rigid.
- (3)  $N$  has reducible complexity.

Then  $\text{G-dim}_R(\text{Tr } M) \leq n$  and  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ .

*Proof.* If  $\text{Tr } M = 0$  then  $\text{G-dim}_R(\text{Tr } M) = 0$  and we have nothing to prove so let  $\text{Tr } M \neq 0$ . As  $\text{Ext}_R^1(M, N) = 0$ ,  $\text{Tor}_1^R(\mathcal{T}_2 M, N) = 0$  by the exact sequence (2.4.2). Since  $N$  is rigid, we have  $\text{Tor}_i^R(\mathcal{T}_2 M, N) = 0$  for all  $i > 0$ . It follows from the exact sequence (2.4.2) again that  $\text{Ext}_R^1(M, R) \otimes_R N = 0$  and since  $N$  is nonzero,  $\text{Ext}_R^1(M, R) = 0$ . Now it is easy to see that  $\mathcal{T}_1 M \approx \Omega \mathcal{T}_2 M$  and so  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ . Therefore we have the following equality.

$$(3.1.1) \quad \text{depth}_R(\text{Tr } M \otimes_R N) + \text{depth } R = \text{depth}_R(\text{Tr } M) + \text{depth}_R(N),$$

by Theorem 2.3. Set  $t = \text{depth}_R(N) - n$ . We argue by induction on  $t$ . If  $t \leq 1$  then  $\text{depth}_R(N) \leq n + 1$ . If  $\text{depth}_R(N) = 0$  then it is clear that  $\text{depth}_R(\text{Tr } M) = \text{depth } R$  by (3.1.1) and so  $\text{G-dim}_R(\text{Tr } M) = 0$  by Auslander-Bridger formula. Now let  $0 < \text{depth}_R(N) \leq n + 1$ . As  $M \approx \text{Tr } \text{Tr } M$ , we obtain the following exact sequence

$$0 \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Tr } M \otimes_R N \rightarrow \text{Hom}_R((\text{Tr } M)^*, N) \rightarrow \text{Ext}_R^2(M, N),$$

from the exact sequence (2.4.1). As  $\text{Ext}_R^1(M, N) = 0$ , we get the following exact sequence.

$$(3.1.2) \quad 0 \rightarrow \text{Tr } M \otimes_R N \rightarrow \text{Hom}_R((\text{Tr } M)^*, N) \rightarrow \text{Ext}_R^2(M, N).$$

Therefore  $\text{Ass}_R(\text{Tr } M \otimes_R N) \subseteq \text{Ass}_R(\text{Hom}_R((\text{Tr } M)^*, N)) \subseteq \text{Ass}_R(N)$ , by the exact sequence (3.1.2). Hence  $\text{depth}_R(\text{Tr } M \otimes_R N) > 0$ . Now by (3.1.1), it is easy to see that  $\text{depth}_R(\text{Tr } M) \geq \text{depth } R - n$  and so  $\text{G-dim}_R(\text{Tr } M) \leq n$ .

Now suppose that  $t > 1$  and consider the following exact sequence

$$(3.1.3) \quad 0 \rightarrow \Omega M \rightarrow F \rightarrow M \rightarrow 0,$$

where  $F$  is a free  $R$ -module. From the exact sequence (3.1.3), we obtain the following exact sequence

$$0 \rightarrow M^* \rightarrow F^* \rightarrow (\Omega M)^* \rightarrow \mathbb{D}(M) \rightarrow \mathbb{D}(F) \rightarrow \mathbb{D}(\Omega M) \rightarrow 0.$$

Where  $\mathbb{D}(X) \approx \text{Tr } X$  for all  $R$ -modules  $X$  by [4, Lemma 3.9]. As  $\text{Ext}_R^1(M, R) = 0$ , we get the following exact sequence

$$(3.1.4) \quad 0 \rightarrow \mathbb{D}(M) \rightarrow \mathbb{D}(F) \rightarrow \mathbb{D}(\Omega M) \rightarrow 0.$$

Note that  $\mathbb{D}(F)$  is free. As  $\text{Ext}_R^i(\Omega M, N) \cong \text{Ext}_R^{i+1}(M, N) = 0$  for all  $1 \leq i \leq \text{depth}_R(N) - n - 1$ , we have  $\text{G-dim}_R(\text{Tr } \Omega M) \leq n + 1$  by induction hypothesis. Therefore,  $\text{G-dim}_R(\text{Tr } M) \leq n$  by the exact sequence (3.1.4).  $\square$

**Theorem 3.2.** *Let  $R$  be a Gorenstein ring and let  $M$  and  $N$  be non-zero  $R$ -modules such that  $N$  has reducible complexity. Assume  $N$  is rigid. Then the following statements hold true.*

- (i) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N)\}$ , then  $\text{G-dim}_R(M) = 0$ .*
- (ii) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{G-dim}_R(M^*) \leq n - 2$ .*
- (iii) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ , then  $\text{G-dim}_R(M) = 0$ .*

*Proof.* (i). By Lemma 3.1,  $\text{G-dim}_R(\text{Tr } M) = 0$  and so  $\text{G-dim}_R(M) = 0$  by [4, Lemma 4.9].

(ii). Note that  $M^* \approx \Omega^2 \text{Tr } M$ . By Lemma 3.1,  $\text{G-dim}_R(\text{Tr } M) \leq n$  and so  $\text{G-dim}_R(M^*) \leq n - 2$ .

(iii). First note that  $\text{G-dim}_R(\text{Tr } M) = \sup\{i \mid \text{Ext}_R^i(\text{Tr } M, R) \neq 0\}$  by [4, Theorem 4.13]. As  $M$  satisfies  $(S_n)$ ,  $\text{Ext}_R^i(\text{Tr } M, R) = 0$  for all  $1 \leq i \leq n$  by [4, Theorem 4.25]. On the other hand,  $\text{G-dim}_R(\text{Tr } M) \leq n$  by Lemma 3.1. Therefore  $\text{G-dim}_R(\text{Tr } M) = 0$  and so  $\text{G-dim}_R(M) = 0$  by [4, Lemmm 4.9].  $\square$

**Corollary 3.3.** *Let  $R$  be a Gorenstein ring of dimension  $d \geq 2$  and Let  $M$  and  $N$  be non-zero  $R$ -modules. Assume that  $\text{pd}_R(N) = 2$  and  $\text{grade}_R(N) > 0$ . Then the following statements hold true.*

- (i) *If  $d \leq 5$  and  $\text{Ext}_R^1(M, N) = 0$  then  $\text{G-dim}_R(M^*) = 0$ .*
- (ii) *Assume that  $M$  satisfies  $(S_n)$  for some  $n \geq 0$  and that  $\text{Ext}_R^i(M, N) = 0$  for some  $i > 0$ . If  $d \leq n + i + 2$  then  $\text{G-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < i$ .*

*Proof.* First note that  $N$  is rigid by [21, Proposition I.4]. Since  $\text{pd}_R(N) = 2$ ,  $\text{depth}_R(N) = d - 2$  by Auslander-Buchsbaum formula.

(i). If  $d \leq 5$  then  $\text{depth}_R(N) - 2 \leq 1$  and the assertion is clear by Theorem 3.2(ii).

(ii). As  $\text{pd}_R(N) < \infty$ ,  $\text{G-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\}$  by [4, Theorem 4.13]. Set  $L = \Omega^{i-1}M$ . Note that  $\text{Ext}_R^1(L, N) \cong \text{Ext}_R^i(M, N) = 0$  and  $L$  satisfies  $(S_{n+i-1})$ . If  $d \leq n + i + 2$  then  $\text{depth}_R(N) \leq n + i$  and so  $\text{G-dim}_R(L) = 0$  by Theorem 3.2(iii). Therefore  $\text{G-dim}_R(M) < i$ .  $\square$

A class of rigid modules was discovered by Peskine and Szpiro [21]. They proved that if  $R$  is local, and the minimal free resolution of  $M$  over  $R$  is of the form

$$(3.1) \quad 0 \rightarrow R^m \rightarrow R^{k+m} \rightarrow R^k \rightarrow 0,$$

for some  $m > 0$  and  $k > 0$ , then  $M$  is rigid. In [23], Tchernev discovered a new class of rigid modules. He showed that if  $R$  is local, and the minimal free resolution of  $M$  over  $R$  is of the form

$$(3.2) \quad 0 \rightarrow R^k \rightarrow R^{m+1} \rightarrow R^m \rightarrow 0,$$

for some  $m > 0$  and  $k > 0$ , then  $M$  is rigid (see [23, Theorem 3.6]).

**Corollary 3.4.** *Let  $R$  be a Gorenstein ring of dimension  $d \geq 2$  and let  $M$  and  $N$  be non-zero  $R$ -modules. If either (3.1) or (3.2) is the minimal free resolution of  $N$  over  $R$  then the following statements hold true.*

- (i) *If  $d \leq 5$  and  $\text{Ext}_R^1(M, N) = 0$  then  $\text{G-dim}_R(M^*) = 0$ .*
- (ii) *Assume that  $M$  satisfies  $(S_n)$  for some  $n \geq 0$  and that  $\text{Ext}_R^i(M, N) = 0$  for some  $i > 0$ . If  $d \leq n + i + 2$  then  $\text{G-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < i$ .*

*Proof.* Since  $\text{pd}_R(N) = 2$ ,  $\text{depth}_R(N) = d - 2$  by Auslander-Buchsbaum formula. As we mentioned before,  $N$  is rigid.

(i). If  $d \leq 5$  then  $\text{depth}_R(N) - 2 \leq 1$  and the assertion is clear by Theorem 3.2(ii).

(ii). As  $\text{pd}_R(N) < \infty$ ,  $\text{G-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\}$  by [4, Theorem 4.13]. Set  $L = \Omega^{i-1}M$ . Note that  $\text{Ext}_R^1(L, N) \cong \text{Ext}_R^i(M, N) = 0$  and  $L$  satisfies  $(S_{n+i-1})$ . If  $d \leq n + i + 2$  then  $\text{depth}_R(N) \leq n + i$  and so  $\text{G-dim}_R(L) = 0$  by Theorem 3.2(iii). Therefore  $\text{G-dim}_R(M) < i$ .  $\square$

In the following, we generalize [18, Corollary 1].

**Corollary 3.5.** *Let  $R$  be a local complete intersection ring and let  $M$  and  $N$  be non-zero  $R$ -modules. Assume the following conditions hold.*

- (i)  *$N$  is rigid.*
- (ii)  *$M$  satisfies  $(S_n)$  for some  $n \geq 0$ .*
- (iii)  *$\text{Ext}_R^i(M, N) = 0$  for some positive integer  $i$  such that  $i \geq \text{depth}_R(N) - n$ .*

*Then  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$ .*

*Proof.* Set  $L = \Omega^{i-1}M$ . Note that  $L$  satisfies  $(S_{n+i-1})$  and  $\text{Ext}_R^1(L, N) = 0$ . Now by Theorem 3.2(iii),  $\text{CI-dim}_R(L) = \text{G-dim}_R(L) = 0$ . By Lemma 3.1,  $\text{Tor}_j^R(\text{Tr } L, N) = 0$  for all  $j > 0$  and so  $\text{Ext}_R^j(L, N) = 0$  for all  $j > 0$  by Theorem 2.7. Therefore  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$ .  $\square$

The following is a generalization of [18, Corollary 2]

**Corollary 3.6.** *Let  $R$  be a local complete intersection ring and let  $M$  and  $N$  be non-zero  $R$ -modules. Suppose that  $N$  is rigid and that  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ . If  $\text{depth}_R(N) - n \leq \text{CI-dim}_R(M)$  then for all  $i > 0$  in the range  $\text{depth}_R(N) - n \leq i \leq \text{CI-dim}_R(M)$ , we have  $\text{Ext}_R^i(M, N) \neq 0$ .*

*Proof.* If  $\text{Ext}_R^i(M, N) = 0$  for some  $\text{depth}_R(N) - n \leq i \leq \text{CI-dim}_R(M)$ , then  $\text{Ext}_R^1(\Omega^{i-1}M, N) \cong \text{Ext}_R^i(M, N) = 0$ . Note that  $\Omega^{i-1}M$  satisfies  $(S_{n+i-1})$ . Now by Theorem 3.2(iii), we have  $\text{CI-dim}_R(\Omega^{i-1}M) = \text{G-dim}_R(\Omega^{i-1}M) = 0$ . Therefore  $\text{CI-dim}_R(M) < i$  by [7, Lemma 1.9], which is a contradiction.  $\square$

The following is a generalization of [19, Theorem 1]. One can also deduce (i),(iii) from [18, Corollary 1] and [5, Proposition 4.10].

**Corollary 3.7.** *Let  $R$  be a regular ring and let  $M$  and  $N$  be non-zero  $R$ -modules. The following statements hold true.*

- (i) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N)\}$ , then  $M$  is free.*
- (ii) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{pd}_R(M^*) \leq n - 2$ .*
- (iii) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ , then  $M$  is free.*

*Proof.* Over a regular local ring, all finitely generated modules are rigid (see [20, corollary 1]). As  $\text{pd}_R(X) < \infty$  for all  $R$ -modules  $X$ ,  $\text{pd}_R(X) = \text{G-dim}_R(X)$  for all  $R$ -modules  $X$ . Therefore the assertion is clear by Theorem 3.2.  $\square$

**Corollary 3.8.** *Let  $R$  be an admissible hypersurface with isolated singularity of dimension  $d > 1$  and let  $M$  and  $N$  be non-zero  $R$ -modules such that  $\dim_R(N) \leq 1$ . If  $\text{Ext}_R^n(M, N) = 0$  for some  $n > 0$ , then  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < n$ . Moreover, either  $\text{pd}_R(M) < \infty$  or  $\text{pd}_R(N) < \infty$ .*

*Proof.* First note that  $N$  is rigid by [12, 2.8, 3.4]. Set  $L = \Omega^{n-1}M$ . Since  $\text{depth}_R(N) \leq 1$  and  $\text{Ext}_R^1(L, N) \cong \text{Ext}_R^n(M, N) = 0$ , we have  $\text{CI-dim}_R(L) = \text{G-dim}_R(L) = 0$  by Theorem 3.2. Note that  $\text{Tor}_i^R(\text{Tr } L, N) = 0$  for all  $i > 0$  by Lemma 3.1 and so  $\text{Ext}_R^i(L, N) = 0$  for all  $i > 0$  by Theorem 2.7. Therefore,  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < n$  by Theorem 2.6 and [7, Lemma 1.9]. As  $\text{Ext}_R^i(M, N) = 0$  for all  $i \gg 0$ , either  $\text{pd}_R(M) < \infty$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

In [11, Proposition 2.5], for a finitely generated  $R$ -module  $N$ , it is shown that  $[N] = 0$  in  $\overline{G}(R)_{\mathbb{Q}}$  if one of the following conditions hold:

- (1)  $N$  has finite length, or  $N$  is a syzygy of some finite length  $R$ -module.
- (2)  $R$  is Artinian.
- (3)  $R$  is a one-dimensional domain.
- (4)  $R$  is a two-dimensional normal domain with torsion class group.

As another application of Theorem 3.2, we have the following.

**Theorem 3.9.** *Let  $R$  be an admissible hypersurface with isolated singularity and let  $M$  and  $N$  be non-zero  $R$ -modules. Assume  $[N] = 0$  in  $\overline{G}(R)_{\mathbb{Q}}$ . The following statements hold true.*

- (i) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N)\}$  then  $\text{CI-dim}_R(M) = 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ . Moreover, either  $M$  is free or  $N$  has finite projective dimension.*
- (ii) *If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{CI-dim}_R(M^*) \leq n - 2$ . Moreover, either  $\text{pd}_R(M^*) \leq n - 2$  or  $N$  has finite projective dimension.*



- (iii) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ , then  $\text{CI-dim}_R(M) = 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ . Moreover, either  $M$  is free or  $N$  has finite projective dimension.

*Proof.* First note that  $N$  is rigid by [12, Corollary 4.2] and so  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$  by Lemma 3.1.

(i). By Theorem 3.2,  $\text{CI-dim}_R(M) = \text{G-dim}_R(M) = 0$ . Therefore  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by Theorem 2.7. Hence, either  $\text{pd}_R(M) = \text{CI-dim}_R(M) = 0$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.

(ii). By Theorem 3.2,  $\text{CI-dim}_R(M^*) = \text{G-dim}_R(M^*) \leq n - 2$ . Since  $M^* \approx \Omega^2 \text{Tr } M$ , we have  $\text{Tor}_i^R(M^*, N) = 0$  for all  $i > 0$ . By Theorem 2.5, either  $\text{pd}_R(M^*) = \text{CI-dim}_R(M^*) \leq n - 2$  or  $\text{pd}_R(N) < \infty$ .

(iii). By Theorem 3.2,  $\text{CI-dim}_R(M) = \text{G-dim}_R(M) = 0$  and so  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by Theorem 2.7. Therefore, either  $M$  is free or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

**Corollary 3.10.** *Let  $R$  be an admissible hypersurface with isolated singularity and let  $M$  and  $N$  be non-zero  $R$ -modules. Assume that  $[N] = 0$  in  $\overline{G}(R)_{\mathbb{Q}}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ . The following statements hold true.*

- (1) If  $\text{Ext}_R^i(M, N) = 0$  for some positive integer  $i$  such that  $i \geq \text{depth}_R(N) - n$  then  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$ .
- (2) If  $\text{depth}_R(N) - n \leq \text{CI-dim}_R(M)$  then for all  $i > 0$  in the range  $\text{depth}_R(N) - n \leq i \leq \text{CI-dim}_R(M)$ , we have  $\text{Ext}_R^i(M, N) \neq 0$ .

*Proof.* By [12, Corollary 4.2],  $N$  is rigid. Now the assertion is clear by Corollary 3.5 and Corollary 3.6.  $\square$

**Theorem 3.11.** *Let  $R$  be an admissible hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules such that  $\text{cx}_R(N) = 1$ . If the minimal free resolution of  $N$  is eventually periodic of period one then the following statements hold true.*

- (i) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \text{depth}_R(N)$  then  $M$  is free.
- (ii) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{pd}_R(M^*) \leq n - 2$ .
- (iii) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ , then  $M$  is free.

*Proof.* First note that  $N$  is rigid by [12, Corollary 5.6] and so  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$  by Lemma 3.1.

(i). By Theorem 3.2,  $\text{CI-dim}_R(M) = \text{G-dim}_R(M) = 0$ . Therefore  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by Theorem 2.7 and so  $\text{pd}_R(M) = \text{CI-dim}_R(M) = 0$  by Theorem 2.5.

(ii). By Theorem 3.2,  $\text{CI-dim}_R(M^*) = \text{G-dim}_R(M^*) \leq n - 2$ . Since  $M^* \approx \Omega^2 \text{Tr } M$ , we have  $\text{Tor}_i^R(M^*, N) = 0$  for all  $i > 0$  and so  $\text{pd}_R(M^*) = \text{CI-dim}_R(M^*) \leq n - 2$  by Theorem 2.5.

(iii). By Theorem 3.2,  $\text{CI-dim}_R(M) = \text{G-dim}_R(M) = 0$  and so  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by Theorem 2.7. Therefore, either  $M$  is free or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

**Corollary 3.12.** *Let  $R$  be an admissible hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules such that  $\text{cx}_R(N) = 1$ . If the minimal free resolution of  $N$  is eventually periodic of period one then the following statements hold true.*

- (i) If  $\text{depth}_R(N) - n \leq \text{CI-dim}_R(M)$  then for all  $i > 0$  in the range  $\text{depth}_R(N) - n \leq i \leq \text{CI-dim}_R(M)$ , we have  $\text{Ext}_R^i(M, N) \neq 0$ .
- (ii) If  $\text{Ext}_R^i(M, N) = 0$  for some positive integer  $i$  such that  $i \geq \text{depth}_R(N) - n$  then  $\text{pd}_R(M) < i$ .

*Proof.* Note that  $N$  is rigid by [12, corollary 5.6] and so (i) is clear by Corollary 3.6.

(ii). By Corollary 3.5,  $\text{Ext}_R^j(M, N) = 0$  for all  $j \geq i$ . Therefore  $\text{pd}_R(M) < \infty$  by Theorem 2.5 and so  $\text{pd}_R(M) < i$ .  $\square$

**Proposition 3.13.** *Let  $(R, \mathfrak{m})$  be a local hypersurface ring such that  $\widehat{R} = S/(f)$  where  $(S, \mathfrak{n})$  is a complete unramified regular local ring and  $f$  is a regular element of  $S$  contained in  $\mathfrak{n}^2$ . Let  $M$  and  $N$  be non-zero  $R$ -modules such that  $\text{pd}_R(N) < \infty$ . The following statements hold true.*

- (i) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N)\}$  then  $\text{CI-dim}_R(M) = 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ .
- (ii) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and some  $n \geq 2$ , then  $\text{CI-dim}_R(M^*) \leq n - 2$ .
- (iii) If  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{1, \text{depth}_R(N) - n\}$  and  $M$  satisfies  $(S_n)$  for some  $n \geq 0$ , then  $\text{CI-dim}_R(M) = 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ .

*Proof.* Note that  $N$  is rigid by [20, Theorem 3] and so  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$  by Lemma 3.1. Now the assertion is clear by Theorem 2.7 and Theorem 3.2.  $\square$

Let  $R$  be a hypersurface and let  $M$  and  $N$  be  $R$ -modules such that  $\text{length}_R(N) < \infty$ . It is well-known that if  $\text{Ext}_R^i(M, N) = 0$  for some  $i > \text{CI-dim}_R(M)$  then  $\text{Ext}_R^n(M, N) = 0$  for all  $n > \text{CI-dim}_R(M)$  (see for example [9, Corollary 3.5]). In special cases, we can remove the condition that  $i > \text{CI-dim}_R(M)$ .

**Corollary 3.14.** *Let  $(R, \mathfrak{m})$  be a local hypersurface ring such that  $\widehat{R} = S/(f)$  where  $(S, \mathfrak{n})$  is a complete unramified regular local ring and  $f$  is a regular element of  $S$  contained in  $\mathfrak{n}^2$ . Let  $M$  and  $N$  be non-zero  $R$ -modules such that  $\text{length}_R(N) < \infty$ . If  $\text{Ext}_R^n(M, N) = 0$  for some  $n \geq 1$ , then the following statements hold true.*

- (i)  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < n$ .
- (ii) either  $\text{pd}_R(M) < \infty$  or  $\text{pd}_R(N) < \infty$ .

*Proof.* First note that  $N$  is rigid by [16, Theorem 2.4]. Set  $L = \Omega^{n-1}M$ . Since  $\text{Ext}_R^1(L, N) \cong \text{Ext}_R^n(M, N) = 0$ ,  $\text{CI-dim}_R(L) = \text{G-dim}(L) = 0$  by Theorem 3.2 and also  $\text{Tor}_i^R(\text{Tr } L, N) = 0$  for all  $i > 0$  by Lemma 3.1. Therefore  $\text{Ext}_R^i(L, N) = 0$  for all  $i > 0$  by Theorem 2.7 and so  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < n$  by [7, Lemma 1.9] and Theorem 2.6. As  $\text{Ext}_R^i(M, N) = 0$  for all  $i \gg 0$ , either  $\text{pd}_R(M) < \infty$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

Let  $R$  be a Gorenstein local ring and let  $M$  be an  $R$ -module. Suppose that  $\{f_1, \dots, f_n\}$  is a minimal generating set for  $M^*$  and that  $f : M \rightarrow R^n$  is the map  $(f_1, \dots, f_n)$ . If  $M$  is torsion-free then we obtain the following exact sequence.

$$(\text{u.p.f.}) \quad 0 \rightarrow M \xrightarrow{f} R^n \rightarrow M_1 \rightarrow 0.$$

Such an exact sequence, obtained from a system of generators of  $M^*$ , is called a universal pushforward of  $M$ . Note that every universal pushforward of  $M$  is dual exact and so  $\text{Ext}_R^1(M_1, R) = 0$ .

**Theorem 3.15.** *Let  $R$  be an admissible hypersurface of dimension 2. Assume further that  $R$  is normal. Let  $M$  and  $N$  be non-zero  $R$ -modules such that  $N$  is torsion-free. If  $\text{Ext}_R^1(M, N) = 0$  then the following statements hold true.*

- (i) *If  $\text{depth}_R(N) = 1$  then  $\text{CI-dim}_R(M) = 0$ .*
- (ii) *If  $M$  is torsion-free then  $\text{CI-dim}_R(M) = 0$ .*

Moreover,  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  and either  $M$  is free or  $N$  has finite projective dimension.

*Proof.* As  $M \approx \text{Tr Tr } M$ , we obtain the following exact sequence

$$(3.15.1) \quad 0 \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Tr } M \otimes_R N \rightarrow \text{Hom}_R((\text{Tr } M)^*, N) \rightarrow \text{Ext}_R^2(M, N)$$

from the exact sequence (2.4.1). Since  $\text{Ext}_R^1(M, N) = 0$ , we get the following exact sequence.

$$(3.15.2) \quad 0 \rightarrow \text{Tr } M \otimes_R N \rightarrow \text{Hom}_R((\text{Tr } M)^*, N) \rightarrow \text{Ext}_R^2(M, N).$$

As  $N$  is torsion-free, it is easy to see that  $\text{Tr } M \otimes_R N$  is torsion-free by the exact sequence (3.15.2). Therefore,  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$  by [12, Proposition 5.2] and so we have the following equality by Theorem 2.3.

$$(3.15.3) \quad \text{depth}_R(\text{Tr } M \otimes_R N) + \text{depth } R = \text{depth}_R(\text{Tr } M) + \text{depth}_R(N).$$

(i). If  $\text{depth}_R(N) = 1$  then it is clear by (3.15.3) that  $\text{Tr } M$  is maximal Cohen-Macaulay. In other words,  $\text{CI-dim}_R(\text{Tr } M) = 0$  and so  $\text{CI-dim}_R(M) = 0$  by [22, Lemma 3.3].

(ii). If  $\text{depth}_R(N) = 1$  then the assertion is followed by (i), so let  $N$  is maximal Cohen-Macaulay. Consider the universal pushforward of  $M$ .

$$(3.15.4) \quad 0 \rightarrow M \rightarrow F \rightarrow M_1 \rightarrow 0,$$

where  $F$  is free. Note that  $\text{Ext}_R^1(M_1, R) = 0$ . From the exact sequence (3.15.4), we conclude that  $\text{Ext}_R^2(M_1, N) \cong \text{Ext}_R^1(M, N) = 0$ . As  $\Omega M_1 \approx \text{Tr } \mathcal{T}_2 M_1$ , we obtain the following exact sequence

$$0 \rightarrow \text{Ext}_R^1(\Omega M_1, N) \rightarrow \mathcal{T}_2 M_1 \otimes_R N \rightarrow \text{Hom}_R((\mathcal{T}_2 M_1)^*, N) \rightarrow \text{Ext}_R^2(\Omega M_1, N),$$

from the exact sequence (2.4.1). Since  $\text{Ext}_R^1(\Omega M_1, N) = \text{Ext}_R^2(M_1, N) = 0$ , we get the following exact sequence.

$$(3.15.5) \quad 0 \rightarrow \mathcal{T}_2 M_1 \otimes_R N \rightarrow \text{Hom}_R((\mathcal{T}_2 M_1)^*, N) \rightarrow \text{Ext}_R^2(\Omega M_1, N),$$

As  $N$  is torsion-free, it is easy to see by the exact sequence (3.15.5) that  $\mathcal{T}_2 M_1 \otimes_R N$  is torsion-free. Therefore  $\text{Tor}_i^R(\mathcal{T}_2 M_1, N) = 0$  for all  $i > 0$  by [12, Proposition 5.2]. It follows that  $\text{Ext}_R^1(M_1, N) = 0$  by the exact sequence (2.4.2). As  $\text{Ext}_R^1(M_1, R) = 0$ , it is easy to see that  $\text{Tr } M_1 \approx \Omega \mathcal{T}_2 M_1$  and so  $\text{Tor}_i^R(\text{Tr } M_1, N) = 0$  for all  $i > 0$ . Therefore, we have the following equality by Theorem 2.3.

$$(3.15.6) \quad \text{depth}_R(\text{Tr } M_1 \otimes_R N) + \text{depth } R = \text{depth}_R(\text{Tr } M_1) + \text{depth}_R(N).$$

Since  $\text{Ext}_R^1(M_1, N) = 0 = \text{Ext}_R^2(M_1, N)$ , we obtain the following isomorphism

$$(3.15.7) \quad \text{Tr } M_1 \otimes_R N \cong \text{Hom}_R((\text{Tr } M_1)^*, N)$$

from the exact sequence (2.4.1). Note that

$$(3.15.8) \quad \text{depth}_R(\text{Hom}_R((\text{Tr } M_1)^*, N)) \geq \min\{2, \text{depth}_R(N)\}.$$

Since  $N$  is maximal Cohen-Macaulay, we conclude by (3.15.7) and (3.15.8) that  $\text{Tr } M_1 \otimes_R N$  is maximal Cohen-Macaulay. Hence  $\text{Tr } M_1$  is maximal Cohen-Macaulay by (3.15.6). In other words,  $\text{CI-dim}_R(\text{Tr } M_1) = 0$  and so  $\text{CI-dim}_R(M_1) = 0$  by [22, Lemma 3.3]. It follows by the exact sequence (3.15.4) that  $\text{CI-dim}_R(M) = 0$ . As  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ ,  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by Theorem 2.7. Therefore, either  $\text{pd}_R(M) = \text{CI-dim}_R(M) = 0$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

As an application of Theorem 3.15, we have the following result.

**Corollary 3.16.** *Let  $R$  be an admissible hypersurface of dimension 2. Assume further that  $R$  is normal. Let  $M$  be a torsion-free  $R$ -module. If  $\text{Ext}_R^1(M, M) = 0$  then  $M$  is free.*

#### 4. VANISHING OF EXT FOR REFLEXIVE MODULES

Following [16], we say that an  $R$ -module  $M$  has constant rank if there is an integer  $n$  such that  $M_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}$  free of rank  $n$  for every  $\mathfrak{p} \in \text{Ass}(R)$ . Let  $R$  be an unramified regular local ring and let  $M$  and  $N$  be non-zero  $R$ -modules. In [3], Auslander proved that if  $M \otimes_R N$  is torsion-free then  $\text{Tor}_i^R(M, N) = 0$  for all  $i > 0$ . In [16], Huneke and Wiegand generalized this result to hypersurfaces with an extra hypothesis. More precisely, they proved the following result.

**Theorem 4.1.** [16, Theorem 2.7] *Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules such that  $N$  has constant rank. If  $M \otimes_R N$  is reflexive then  $\text{Tor}_i^R(M, N) = 0$  for all  $i > 0$ .*

In the following, we estimate the complete intersection of  $\text{Tr } M$ , in terms of vanishing of the cohomology modules,  $\text{Ext}_R^i(M, N)$ .

**Lemma 4.2.** Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules. Suppose that  $n \geq 0$  is an integer and that the following conditions hold.

- (i)  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N) - n\}$ .
- (ii)  $N$  satisfies  $(S_2)$ .
- (iii)  $N$  has constant rank.

Then  $\text{CI-dim}_R(\text{Tr } M) \leq n$  and  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ .

*Proof.* If  $\text{Tr } M = 0$  then  $\text{CI-dim}_R(\text{Tr } M) = 0$  and we have nothing to prove so let  $\text{Tr } M \neq 0$ . As  $M \approx \text{Tr } \text{Tr } M$ , we obtain the following exact sequence

$$0 \rightarrow \text{Ext}_R^1(M, N) \rightarrow \text{Tr } M \otimes_R N \rightarrow \text{Hom}_R((\text{Tr } M)^*, N) \rightarrow \text{Ext}_R^2(M, N),$$

from the exact sequence (2.4.1). By (i),  $\text{Ext}_R^1(M, N) = 0 = \text{Ext}_R^2(M, N)$  and we get the following isomorphism.

$$(4.2.1) \quad \text{Tr } M \otimes_R N \cong \text{Hom}_R((\text{Tr } M)^*, N).$$

Since  $N$  satisfies  $(S_2)$ , it is easy to see that  $\text{Tr } M \otimes_R N$  satisfies  $(S_2)$  by (4.2.1). Therefore  $\text{Tr } M \otimes_R N$  is reflexive by [14, Theorem 3.6] and so  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$  by Theorem 4.1 and we have the following equality by [16, Proposition 2.5].

$$(4.2.2) \quad \text{depth}_R(\text{Tr } M \otimes_R N) + \text{depth } R = \text{depth}_R(\text{Tr } M) + \text{depth}_R(N).$$

Set  $t = \text{depth}_R(N) - n$ . We argue by induction on  $t$ . If  $t \leq 2$  then  $\text{depth}_R(N) \leq n+2$ . From (4.2.1), we conclude that  $\text{depth}_R(\text{Tr } M \otimes_R N) \geq \min\{2, \text{depth } N\}$ . Now it is easy to see that  $\text{depth}_R(\text{Tr } M) \geq \text{depth } R - n$  by (4.2.2) and so  $\text{CI-dim}_R(\text{Tr } M) \leq n$  by [7, Theorem 1.4].

Now suppose that  $t > 2$ . As  $\Omega M \approx \text{Tr } \mathcal{T}_2 M$ , we obtain the following exact sequence

$$(4.2.3) \quad 0 \rightarrow \text{Ext}_R^1(\Omega M, N) \rightarrow \mathcal{T}_2 M \otimes_R N \rightarrow \text{Hom}_R((\mathcal{T}_2 M)^*, N) \rightarrow \text{Ext}_R^2(\Omega M, N),$$

from the exact sequence (2.4.1). As  $\text{Ext}_R^i(\Omega M, N) \cong \text{Ext}_R^{i+1}(M, N) = 0$  for  $1 \leq i \leq 2$ , we get the following isomorphism

$$(4.2.4) \quad \mathcal{T}_2 M \otimes_R N \cong \text{Hom}_R((\mathcal{T}_2 M)^*, N)$$

If  $\mathcal{T}_2 M = 0$  then  $\Omega M$  is free and so  $\text{pd}_R(M) \leq 1$ . Since  $\text{Ext}_R^1(M, N) = 0$ , we conclude that  $M$  is free by [5, Proposition 4.10] and we have nothing to prove so let  $\mathcal{T}_2 M \neq 0$ . Since  $N$  satisfies  $(S_2)$ , it is easy to see that  $\mathcal{T}_2 M \otimes_R N$  satisfies  $(S_2)$  by (4.2.4). Therefore  $\mathcal{T}_2 M \otimes_R N$  is reflexive by [14, Theorem 3.6] and so  $\text{Tor}_i^R(\mathcal{T}_2 M, N) = 0$  for all  $i > 0$  by Theorem 4.1. It follows by the exact sequence (2.4.2) that  $\text{Ext}_R^1(M, R) \otimes_R N = 0$ . As  $N$  is nonzero,  $\text{Ext}_R^1(M, R) = 0$ . Now consider the following exact sequence

$$(4.2.5) \quad 0 \rightarrow \Omega M \rightarrow F \rightarrow M \rightarrow 0,$$

where  $F$  is a free  $R$ -module. From the exact sequence (4.2.5), we obtain the following exact sequence

$$0 \rightarrow M^* \rightarrow F^* \rightarrow (\Omega M)^* \rightarrow \mathbb{D}(M) \rightarrow \mathbb{D}(F) \rightarrow \mathbb{D}(\Omega M) \rightarrow 0.$$

Where  $\mathbb{D}(X) \approx \text{Tr } X$  for all  $R$ -modules  $X$  by [4, Lemma 3.9]. As  $\text{Ext}_R^1(M, R) = 0$ , we get the following exact sequence

$$(4.2.5) \quad 0 \rightarrow \mathbb{D}(M) \rightarrow \mathbb{D}(F) \rightarrow \mathbb{D}(\Omega M) \rightarrow 0.$$

Note that  $\mathbb{D}(F)$  is free. As  $\text{Ext}_R^i(\Omega M, N) \cong \text{Ext}_R^{i+1}(M, N) = 0$  for  $1 \leq i \leq \text{depth}_R(N) - n - 1$ , we have  $\text{CI-dim}_R(\text{Tr } \Omega M) \leq n + 1$  by induction hypothesis. Therefore,  $\text{CI-dim}_R(\text{Tr } M) \leq n$  by the exact sequence (4.2.5).  $\square$

**Theorem 4.3.** *Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules. Suppose that  $n \geq 0$  is an integer and that the following conditions hold.*

- (i)  $\text{Ext}_R^i(M, N) = 0$  for all  $n + 1 \leq i \leq n + \max\{\text{depth}_R(N), 2\}$ .
- (ii)  $N$  satisfies  $(S_2)$ .
- (iii)  $N$  has constant rank.

*Then  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} \leq n$ .*

*Proof.* Set  $L = \Omega^n M$ . Since  $\text{Ext}_R^i(L, N) \cong \text{Ext}_R^{i+n}(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N)\}$ , we have  $\text{CI-dim}_R(\text{Tr } L) = 0$  by Lemma 4.2 and so  $\text{CI-dim}_R(L) = 0$  by [22, Lemma 3.3]. Hence  $\text{CI-dim}_R(M) \leq n$  by [7, Lemma 1.9]. Note that  $\text{cx}_R(M) \leq 1$ . Therefore  $\text{Ext}_R^i(M, N) = 0$  for all  $i > \text{CI-dim}_R(M)$  by [17, Corollary 2.6] and  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\}$  by Theorem 2.6.  $\square$

**Theorem 4.4.** *Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules. Assume the following conditions hold.*

- (i)  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N) - 2\}$ .
- (ii)  $N$  satisfies  $(S_2)$ .

(iii)  $N$  has constant rank.

Then  $\text{CI-dim}_R(M^*) = 0$ . Moreover, either  $M^*$  is free or  $\text{pd}_R(N) < \infty$ .

*Proof.* By Lemma 4.2,  $\text{CI-dim}_R(\text{Tr } M) \leq 2$  and  $\text{Tor}_i^R(\text{Tr } M, N) = 0$  for all  $i > 0$ . As  $M^* \approx \Omega^2 \text{Tr } M$ ,  $\text{CI-dim}_R(M^*) = 0$  and  $\text{Tor}_i^R(M^*, N) = 0$  for all  $i > 0$ . Therefore either  $\text{pd}_R(M^*) = \text{CI-dim}_R(M^*) = 0$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

**Corollary 4.5.** *Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules. Suppose that  $n \geq 0$  is an integer and that the following conditions hold.*

- (i)  $\text{Ext}_R^i(M, N) = 0$  for all  $1 \leq i \leq \max\{2, \text{depth}_R(N) - n\}$ .
- (ii)  $N$  satisfies  $(S_2)$ .
- (iii)  $N$  has constant rank.
- (iv)  $M$  satisfies  $(S_n)$ .

Then  $\text{CI-dim}_R(M) = 0$  and  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$ . Moreover either  $M$  is free or  $\text{pd}_R(N) < \infty$ .

*Proof.* Since  $M$  satisfies  $(S_n)$ , we have

$$(4.5.1) \quad \text{Ext}_R^i(\text{Tr } M, R) = 0 \text{ for all } 1 \leq i \leq n,$$

by [4, Theorem 4.25]. Note that

$$(4.5.2) \quad \text{CI-dim}_R(\text{Tr } M) = \text{G-dim}_R(\text{Tr } M) = \sup\{i \mid \text{Ext}_R^i(\text{Tr } M, R) \neq 0\},$$

by [4, Theorem 4.13] and [7, Theorem 1.4]. Also by Lemma 4.2,

$$(4.5.3) \quad \text{CI-dim}_R(\text{Tr } M) \leq n.$$

Now by (4.5.1), (4.5.2) and (4.5.3) it is clear that  $\text{CI-dim}_R(\text{Tr } M) = 0$  and so  $\text{CI-dim}_R(M) = 0$  by [22, Lemma 3.3]. As  $\text{cx}_R(M) \leq 1$ ,  $\text{Ext}_R^i(M, N) = 0$  for all  $i > 0$  by [17, Corollary 2.6] and so either  $\text{pd}_R(M) = \text{CI-dim}_R(M) = 0$  or  $\text{pd}_R(N) < \infty$  by Theorem 2.5.  $\square$

**Corollary 4.6.** *Let  $R$  be a hypersurface and let  $M$  and  $N$  be non-zero  $R$ -modules. Suppose that  $t > 0$  and  $n \geq 0$  are integers and that the following conditions hold.*

- (i)  $\text{Ext}_R^t(M, N) = \text{Ext}_R^{t+1}(M, N) = 0$ .
- (ii)  $N$  satisfies  $(S_2)$ .
- (iii)  $N$  has constant rank.
- (iv)  $M$  satisfies  $(S_n)$ .
- (v)  $\text{depth}_R(N) \leq n + t + 1$ .

Then  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < t$ .

*Proof.* Set  $L = \Omega^{t-1}M$ . Note that  $L$  satisfies  $(S_{n+t-1})$  and also  $\text{Ext}_R^1(L, N) = \text{Ext}_R^2(L, N) = 0$ . Therefore  $\text{CI-dim}_R(L) = 0$  and  $\text{Ext}_R^i(L, N) = 0$  for all  $i > 0$  by Corollary 4.5 and so  $\text{CI-dim}_R(M) = \sup\{i \mid \text{Ext}_R^i(M, N) \neq 0\} < t$  by [7, Lemma 1.9] and Theorem 2.6.  $\square$

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## REFERENCES

1. T. Araya and Y. Yoshino, *Remarks on a depth formula, a grade inequality and a conjecture of Auslander*, Comm. Algebra 26 (1998), 3793-3806. 5
2. M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, in: Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, vol. 1966/67, Secrétariat mathématique, Paris, 1967. 3
3. M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. 5 (1961), 631-647. 1, 4, 12
4. M. Auslander and M. Bridger, *Stable module theory*, Mem. of the AMS 94, Amer. Math. Soc., Providence 1969. 3, 4, 5, 6, 7, 13, 14
5. M. Auslander and O. Goldman, *Maximal Orders*, Trans. Amer. Math. Soc. 97 (1960), 1-24. 8, 13
6. L. L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*. Invent. Math. 142 (2000): 285-318. 5
7. L. L. Avramov, V. N. Gasharov and I. V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. 86 (1997), 67-114. 3, 4, 8, 10, 13, 14
8. P. Bergh, *Modules with reducible complexity*, J. Algebra 310 (2007), 132-147. 3
9. P. Bergh, *On the vanishing of homology with modules of finite length*, Math. Scand. to appear. 10
10. P. Bergh and D. Jorgensen, *The depth formula for modules with reducible complexity*. Illinois J. Math. to appear. 3, 4
11. O. Celikbas and H. Dao. *Asymptotic behavior of Ext functors for modules of finite complete intersection dimension*, Math. Z., Volume 269, Issue 3-4 (2011), pp 1005-1020. 8
12. H. Dao, *Decency and Tor-rigidity for modules over hypersurfaces*, Transactions of the AMS, to appear. 8, 9, 10, 11
13. H. Dao, *Asymptotic behavior of Tor over complete intersections and applications*, arXiv math.AC/ 0710.5818, preprint.
14. E. G. Evans and P. Griffith. *Syzygies*, volume 106 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1985. 5, 12, 13
15. T. H. Gulliksen, *A change of rings theorem, with applications to Poincare series and intersection multiplicity*, Math. Scand. 34 (1974), 167-183. 3
16. C. Huneke and R. Wiegand, *Tensor products of modules and the rigidity of Tor*, Math. Ann. 299 (1994), 449-476. 4, 10, 12
17. D. Jorgensen, *Vanishing of (co)homology over commutative rings*, Comm. Alg. 29 (2001), 1883-1898. 13, 14
18. P. Jothilingam, *Syzygies and Ext*. Math. Z. 188 (1985), 278-282. 1, 7, 8
19. P. Jothilingam and T. Duraivel, *Test Modules for Projectivity of Duals*, Comm. Alg. 38:8 (2010), 2762-2767. 1, 2, 8
20. S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Ill. J. Math. 10 (1966), 220-226. 1, 8, 10
21. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. 42 (1973), 471-119. 6, 7
22. A. Sadeghi, *A note on the depth formula and vanishing of cohomology*. preprint, 2012. 5, 11, 12, 13, 14
23. A. Tchernev, *Free direct summands of maximal rank and rigidity in projective dimension two*. Comm. Alg. 34:2 (2006), 671-679. 7

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